K22P 0189

Reg. No. : ....

Name : ....

II Semester M.Sc. Degree (C.B.S.S. – Reg./Supple./Imp.) Examination, April 2022
(2018 Admission Onwards)
MATHEMATICS
MAT 2C 06 – Advanced Abstract Algebra

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions. **Each** question carries **4** marks.

1. Prove that $\mathbb{Z}[i]$ is an Euclidean domain.
2. Construct a field of four elements by showing $x^2 + x + 1$ is irreducible in $\mathbb{Z}_2[x]$.
3. Show that it is not always possible to construct with straight edge and compass, the side of a cube that has double the volume of original cube.
4. Show that if F is a finite field of characteristic p, then the map $\sigma_p : F \to F$ defined by $\sigma_p(a) = a^p$, for $a \in F$, is an automorphism.
5. Prove that there exists only an unique algebraic closure of a field up to isomorphism.
6. If E is a finite extension of F, Then prove that $\{E : F\}$ divides $[E : F]$. **(4×4=16)**

PART – B

Answer **any 4** questions without omitting any Unit. **Each** question carries **16** marks.

UNIT – I

7. a) State and prove Kronecker's theorem. **8**

   b) Prove that $\mathbb{Q}(\pi) \cong \mathbb{Q}(x)$, where $\mathbb{Q}(x)$ is the field of rational numbers over $\mathbb{Q}$. **4**

   c) Prove that $\mathbb{R}[x]/< x^2+1 > \cong \mathbb{R}(i) \cong \mathbb{C}$. **4**

8. a) Prove that if D is a UFD, then D[x] is a UFD. **8**

   b) Show that not every UFD is a PID. **3**

   c) Express $18x^2 - 12x + 48$ in $\mathbb{Q}[x]$ as a product of its content with a primitive polynomial. **5**

P.T.O.

9. a) Prove that for a Euclidean domain with Euclidean norm v, v(1) is minimal among all v (a) for non-zero $a \in D$, and also $u \in D$ is a unit if and only if, $v(u) = v(1)$.

b) Let p the an odd prime in $\mathbb{Z}$. Then prove that $p = a^2 + b^2$ for $a, b \in \mathbb{Z}$, if and only if $p \equiv 1 \pmod{p}$.

UNIT – II

10. a) Prove that there exists finite of p^n elements for every prime power p^n.

b) Let p be a prime and $n \in \mathbb{Z}^+$. Prove that if E and E′ are fields of order p^n, then $E \cong E'$.

11. a) Find the degree and basis for $\mathbb{Q}\left(\sqrt[3]{5}, 2\right)$ and $\mathbb{Q}\left(\sqrt{2}+\sqrt{3}\right)$ over $\mathbb{Q}$.

b) Prove in detail that $\mathbb{Q}\left(\sqrt{3}+\sqrt{7}\right) = \mathbb{Q}\left(\sqrt{3}, \sqrt{7}\right)$.

c) Define algebraic closure of a field and prove that, a field F is algebraically closed if and only if, every non constant polynomial in F[x] factors in F[x] into linear factors.

12. a) Describe the group $G\left(\mathbb{Q}\sqrt{2}, \sqrt{3} / \mathbb{Q}\right)$.

b) Let F be a field and let α, β are algebraic over F. Then prove that $F(\alpha) \cong F(\beta)$ if and only if α and β are conjugates over F.

c) Let $\{\sigma_i / i \in I\}$ be the collection of automorphisms of a field $\bar{F}$. Then prove that the set $E_{\{\sigma_i\}}$ of all $a \in E$ left fixed by every σ_i for $i \in I$, forms a subfield of E.

UNIT – III

13. a) Prove that a finite separable extension of a field is a simple extension.

b) Every finite field is perfect.

14. a) Show that $[E : F] = 2$, then E is splitting field over F.

b) Show that if $E \leq \bar{F}$, is a splitting field over F, then every irreducible polynomial in F[x] having a zero in E splits in E.

c) Find the splitting field and its degree over $\mathbb{Q}$ of the polynomial $(x^2 - 2)(x^3 - 2)$ in $\mathbb{Q}[x]$.

15. a) Let K be a finite extension of degree n of a finite field F of p^r elements. Then G(K/F) is cyclic of order n and is generated by σ_{p^r}, for $\alpha \in K$, $\sigma_{p^r}(\alpha) = \alpha^{p^r}$.

b) State isomorphism extension theorem.

c) Let f(x) be irreducible in F[x]. Then prove that all zeros in f(x) in $\bar{F}$ has same multiplicity.

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